BMath-Differential Geometry-II, Supplementary Exam

INSTRUCTIONS: All problems carry equal weight and are compulsory. Total time 3 hours. Please use notations and terminology as in the course, use results done in the class without proving them. If you use a problem from some assignment/quiz/homework, please provide its solution.

1. Let M be a smooth manifold and $f, g \in C^\infty(M)$. Prove that $\mathrm{d}(fg) = g\mathrm{d}(f) + f\mathrm{d}(g)$.

2. Compute
   (i) $f^*(\omega)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (xy - yz, xyz, 0)$ and $\omega = x\mathrm{d}(y) \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$.
   (ii) $\mathrm{d}(f^*\omega)$ where $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (-x, x)$ and $\omega = \mathrm{d}y - \mathrm{d}x$. (5+5)

3. Compute the Lie algebras of the real Lie groups $SU(n)$, $SO(n)$ and their respective dimensions. Prove that these are compact. (5+5)

4. Let G be a connected Lie group. Prove that the kernel of $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$ equals the center of G. What is the Lie algebra of $\ker(\mathrm{Ad})$? Explain. (5+5)

5. **(i)** Let M be a smooth manifold admiting an atlas consisting of exactly two charts $\{(U, x),\ (V, y)\}$ with $U \cap V$ (nonempty) connected. Prove that M is orientable. Deduce that S^n is orientable for $n \geq 2$.
   (ii) Prove that S^1 is orientable. (4+1+5)